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Constructing Fuzzy Subgroups of Symmetric Groups S_4

R. Sulaiman

Department of Mathematics, Faculty of Mathematics and Sciences
Universitas Negeri Surabaya, Surabaya 60231 Indonesia
sulaimanraden@yahoo.com

Abstract

This article computes the number of fuzzy subgroups of symmetric group S_4 and constructs some of them. First, an equivalence relation on the set of all fuzzy subgroups of a group G is defined. The diagram of subgroups lattice of S_4 is used to determine the number of fuzzy subgroups of S_4. We find the total number of fuzzy subgroups of S_4 is 220.

Mathematics Subject Classification: 20B30, 20B35, 03G10

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1 Introduction

After the first paper by Zadeh in 1965 [3], several aspects of fuzzy subset were studied in the first fifteen years. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups by Rosenfeld in 1971 [1]. Without any equivalence relation on fuzzy subgroups of group G, the number of fuzzy subgroups is infinite, even for the trivial group $\{e\}$. Some authors have used the equivalence relation of fuzzy sets to study the equivalence of fuzzy subgroups ([2], [4], [5], [9], [10], [11]). All of them have treated the particular case of finite abelian group. It is interesting to count the number of fuzzy subgroups of nonabelian groups and construct them. Laszlo in [2] has studied about the construction of fuzzy subgroup of group of order one to six. In the first paper of this topic, Sulaiman and Abd Ghafur [6] have counted the number of fuzzy subgroups of nonabelian symmetric groups S_2, S_3 and alternating group A_4. In the other paper, they [7] have counted the number of fuzzy subgroups of group defined by a presentation.

In [8] we have constructed the diagram of subgroups lattice of symmetric group S_4. That result is very important for this paper, in order to count the number of fuzzy subgroup of S_4.

2 Preliminary Notes

Definition 2.1 *Let G be a group. A function μ from G into [0,1] is called a fuzzy subgroup of G if $\mu(xy) \geq min\{\mu(x), \mu(y)\}$, $\forall x, y \in G$ and $\mu(x^{-1}) \geq \mu(x)$, $\forall x \in G$.*

Theorem 2.2 *Sulaiman and Abd Gahfur [6]. Function $\mu : G \to [0,1]$ is a fuzzy subgroup of G if there is a chain $P_1 < P_2 < ... < P_n = G$ in subgroups lattice of G such that μ can be written as*

$$\mu(x) = \begin{cases} \theta_1, & x \in P_1 \\ \theta_2, & x \in P_2 \setminus P_1 \\ \vdots & \\ \theta_n, & x \in P_n \setminus P_{n-1} \end{cases} \tag{1}$$

Without any equivalence relations on fuzzy subgroups of a group G, the number of fuzzy subgroups is infinite even for the trivial group $\{e\}$. So we define equivalence relation on the set of all fuzzy subgroups of a given group. Using this equivalent, we have constructed all of the fuzzy subgroups of symmetric groups S_4.

Definition 2.3 *Let μ, γ be fuzzy subgroups of G of the form*

$$\mu(x) = \begin{cases} \theta_1, & x \in P_1 \\ \theta_2, & x \in P_2 \setminus P_1 \\ \vdots & \\ \theta_n, & x \in P_n \setminus P_{n-1} \end{cases}, \gamma(x) = \begin{cases} \delta_1, & x \in M_1 \\ \delta_2, & x \in M_2 \setminus M_1 \\ \vdots & \\ \delta_n, & x \in M_m \setminus M_{m-1} \end{cases}.$$

Then we say that μ and γ are equivalent and write $\mu \sim \gamma$, if (1)$m = n$ and (2)$P_i = M_i, \forall i \in \{1, 2, ..., m\}$.
It is easily checked that this relation is indeed an equivalence relation. Two fuzzy subgroups of G are said to be different if they are not equivalent.

Example 2.4 *Consider the group $G = Z_{12}$. Let $\mu, \gamma, \alpha, \beta$ be functions from Z_{12} into $[0,1]$ such that*

$$\mu(x) = \begin{cases} 1, & x \in \{0, 2, 4, 6, 8, 10\} \\ \frac{1}{2}, & x \in \{1, 3, 5, 7, 9, 11\} \end{cases}, \gamma(x) = \begin{cases} 1, & x \in \{0, 1, 2, 3, 4, 5\} \\ \frac{1}{2}, & x \in \{6, 7, 8, 9, 10, 11\} \end{cases}$$

$$\alpha(x) = \begin{cases} 1, & x \in \{0,4,8\} \\ \frac{1}{2}, & x \in \{2,6,10\} \\ \frac{1}{3}, & x \in \{1,3,5,7,9,11\} \end{cases}, \ \beta(x) = \begin{cases} \frac{1}{2}, & x \in \{0,4,8\} \\ \frac{1}{4}, & x \in \{2,6,10\} \\ 0, & x \in \{1,3,5,7,9,11\} \end{cases}.$$

Note that $P_1(\mu) = \{0,2,4,6,8,10\}$, $P_2(\mu) = Z_{12}$. Thus $P_1(\mu), P_2(\mu)$ are both subgroups of Z_{12}. By Theorem 2.2, μ is a fuzzy subgroup of Z_{12}. Similarly, we can show that α and β both are fuzzy subgroups of Z_{12}. Note that $P_1(\gamma)$ is not a subgroup of Z_{12}. Hence γ is not a fuzzy subgroup of Z_{12}. Since $|\mu| \neq |\alpha|$ and $|\mu| \neq |\beta|$,by Definition 2.3 we have μ is not equivalent with α and μ are not equivalent with β. Note that $|\alpha| = |\beta|$ and it is easy to show that $P_i(\alpha) = P_i(\beta), \forall i \in \{1,2,3\}$. Thus $\alpha \sim \beta$.

Lemma 2.5 *The number of fuzzy subgroups of G is equal to the number of chain on the subgroups lattice of G.*
Proof: Using Theorem 2.2 and Definition 2.3.

3 Constructing fuzzy subgroups of S_4

Let $S_4 = \{i = id, \sigma_1, \sigma_2, ..., \sigma_9, \tau_1, \tau_2, ..., \tau_8, \alpha_1, \alpha_2, ..., \alpha_6\}$ with $\sigma_1 = (1234)$,$\sigma_2 = (13)(24)$, $\sigma_3 = (1432)$, $\sigma_4 = (1243)$,$\sigma_5 = (14)(23)$, $\sigma_6 = (1342)$, $\sigma_7 = (1324)$, $\sigma_8 = (12)(34)$, $\sigma_9 = (1432)$,$\tau_1 = (234)$,$\tau_2 = (243)$, $\tau_3 = (134)$, $\tau_4 = (143)$, $\tau_5 = (124)$, $\tau_6 = (142)$, $\tau_7 = (123)$, $\tau_8 = (132)$, $\alpha_1 = (12)$,$\alpha_1 = (12)$,$\alpha_2 = (13)$, $\alpha_3 = (14)$, $\alpha_4 = (23)$, $\alpha_5 = (24)$, $\alpha_6 = (34)$.

We have got thirty subgroups of S_4 (see in [8]) those are, $K_1 = \{i\}$,$K_2 = \{i, \sigma_2\}$, $K_3 = \{i, \sigma_5\}$, $K_4 = \{i, \sigma_8\}$, $K_5 = \{i, \alpha_1\}$, $K_6 = \{i, \alpha_2\}$, $K_7 = \{i, \alpha_3\}$, $K_8 = \{i, \alpha_4\}$, $K_9 = \{i, \alpha_5\}$, $K_{10} = \{i, \alpha_6\}$,$L_{11} = \{i, \tau_1, \tau_2\}$, $L_{12} = \{i, \tau_3, \tau_4\}$, $L_{13} = \{i, \tau_5, \tau_6\}$, $L_{14} = \{i, \tau_7, \tau_8\}$, $M_{15} = \{i, \sigma_1, \sigma_2, \sigma_3\}$,$M_{16} = \{i, \sigma_4, \sigma_5, \sigma_6\}$, $M_{17} = \{i, \sigma_7, \sigma_8, \sigma_9\}$,$M_{18} = \{i, \sigma_2, \sigma_5, \sigma_8\}$, $M_{19} = \{i, \sigma_2, \alpha_2, \alpha_5\}$,
$M_{20} = \{i, \sigma_5, \alpha_3, \alpha_4\}$, $M_{21} = \{i, \sigma_8, \alpha_1, \alpha_6\}$,$N_{22} = \{i, \tau_1, \tau_2, \alpha_4, \alpha_5, \alpha_6\}$, $N_{23} = \{i, \tau_3, \tau_4, \alpha_2, \alpha_3, \alpha_6\}$, $N_{24} = \{i, \tau_5, \tau_6, \alpha_1, \alpha_3, \alpha_5\}$, $N_{25} = \{i, \tau_7, \tau_8, \alpha_1, \alpha_2, \alpha_4\}$, $P_{26} = \{i, \sigma_1, \sigma_2, \sigma_3, \sigma_5, \sigma_8, \alpha_2, \alpha_5\}$, $P_{27} = \{i, \sigma_2, \sigma_4, \sigma_5, \sigma_6, \sigma_8, \alpha_3, \alpha_4\}$, $P_{28} = \{i, \sigma_2, \sigma_5, \sigma_7, \sigma_8, \sigma_9, \alpha_1, \alpha_6\}$, alternating group A_4 and S_4 itself.

The diagram of subgroups lattice of S_4 is shown as fig.1. We will count the number of the fuzzy subgroups of S_4 by observing that diagram and using Lemma 1. We can see that the maximal chain on that lattice consists of five subgroups of S_4. Therefore, the fuzzy subgroup μ of S_4 has length 1,2,3,4 or 5. Let μ be a fuzzy subgroup of S_4. We will identify μ according to $P_1(\mu)$. Every subgroup of S_4 can be chosen to be $P_1(\mu)$.

fig.1

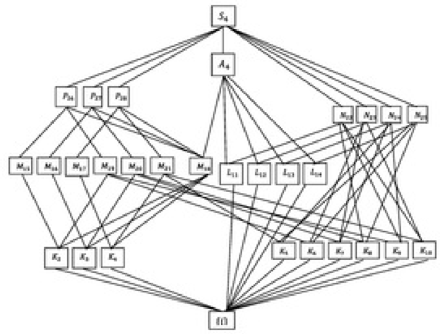

Figure 1: Subgroup lattice of S_4.

If $P_1(\mu) = S_4$, we only have one fuzzy subgroup of S_4, that is $\mu_1(x) = \theta_1, \forall x \in S_4$. If $P_1(\mu) = A_4$, then the only option we have is $S_4 = P_2(\mu)$. Therefore we only have one fuzzy subgroup of S_4 with $P_1(\mu) = A_4$ that is

$$\mu_2(x) = \begin{cases} \theta_1, & \text{x} \in A_4 \\ \theta_2, & \text{x} \in S_4 \backslash A_4 \end{cases} .$$

Similarly, we have one fuzzy subgroup for $P_1(\mu) = P_{26}, P_1(\mu) = P_{27}, P_1(\mu) = P_{28}, P_1(\mu) = N_{22}, P_1(\mu) = N_{23}, P_1(\mu) = N_{24}, P_1(\mu) = N_{25}$. If $P_1(\mu) = M_{15}$, then we have two chains, those are $M_{15} < S_4$ and $M_{15} < P_{26} < S_4$. Therefore, we get two fuzzy subgroups of S_4 with $P_1 = M_{15}$, those are

$$\mu_{10}(x) = \begin{cases} \theta_1, & \text{x} \in M_{15} \\ \theta_2, & \text{x} \in S_4 \backslash M_{15} \end{cases} \quad \text{and } \mu_{11}(x) = \begin{cases} \theta_1, & \text{x} \in M_{15} \\ \theta_2, & \text{x} \in P_{26} \backslash M_{15} \\ \theta_3, & \text{x} \in S_4 \backslash P_{26} \end{cases}$$

Similarly, we have two fuzzy subgroups for $P_1\mu = M_{16}, P_1\mu = M_{17}, P_1\mu = M_{19}, P_1\mu = M_{20}, P_1\mu = M_{21}$. If $P_1 = M_{18}$, then we have five chains, those are $M_{18} < P_{26} < S_4$, $M_{18} < P_{27} < S_4$,$M_{18} < P_{28} < S_4$ and $M_{18} < A_4 < S_4$. Therefore, we get five fuzzy subgroups of S_4 with $P_1 = M_{18}$, those are

$$\mu_{22}(x) = \begin{cases} \theta_1, & \text{x} \in M_{18} \\ \theta_2, & \text{x} \in S_4 \backslash M_{18} \end{cases}, \mu_{23}(x) = \begin{cases} \theta_1, & \text{x} \in M_{18} \\ \theta_2, & \text{x} \in P_{26} \backslash M_{18} \\ \theta_3, & \text{x} \in S_4 \backslash P_{26} \end{cases}$$

$$\mu_{24}(x) = \begin{cases} \theta_1, & \text{x} \in M_{18} \\ \theta_2, & \text{x} \in P_{27} \backslash M_{18} \\ \theta_3, & \text{x} \in S_4 \backslash P_{27} \end{cases}, \mu_{25}(x) = \begin{cases} \theta_1, & \text{x} \in M_{18} \\ \theta_2, & \text{x} \in P_{28} \backslash M_{18} \\ \theta_3, & \text{x} \in S_4 \backslash P_{28} \end{cases} \quad \textit{and}$$

$$\mu_{26}(x) = \begin{cases} \theta_1, & \text{x} \in M_{18} \\ \theta_2, & \text{x} \in A_4 \backslash M_{18} \\ \theta_3, & \text{x} \in S_4 \backslash A_4 \end{cases}.$$

If $P_1(\mu) = L_{11}$, then we have three chains, those are $L_{11} < S_4$, $L_{11} < A_4 < S_4$ and $L_{11} < N_{22} < S_4$. Therefore, we get three fuzzy subgroups of S_4 with $P_1(\mu) = L_{11}$ and we have the same number for $P_1(\mu) = L_{12}$, $P_1(\mu) = L_{13}$,$P_1(\mu) = L_{14}$. By similar method, we have
(1) Twelve fuzzy subgroups of S_4 for $P_1()\mu = K_2, P_1()\mu = K_3$ and $P_1 = K_4$.
(2) Six fuzzy subgroups of S_4 for $P_1(\mu) = K_5$, $P_1(\mu) = K_6$, $P_1(\mu) = K_7$, $P_1(\mu) = K_8$, $P_1(\mu) = K_9$ and $P_1(\mu) = K_{10}$.
(3) For $P_1(\mu) = \{e\}$ we have 109 subgroups fuzzy. Thus, the total number of fuzzy subgroups of S_4 is $1 + (81) + (62) + 5 + (43) + (312) + (66) + 109 = 220$.

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